ON RADIATIVE HEAT TRANSFER IN A PLANE LAYER OF AN ABSORBING MEDIUM

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Recently there has arisen increased interest in the study of radiative heat transfer between geometrically simple systems, both as autonomous problems and as elements of more complex problems.

Problems of this kind have been treated by many authors [1–11], who have considered gray, diffusely emitting and absorbing boundaries and gray nonscattering media. In most cases these investigations were restricted either to the derivation of approximate formulas for the net radiative flux, without an exact analysis of the temperature distribution in the layer [5–7], or to numerical computation [1–4]. In the latter case, with the exception of [8], which contains a numerical analysis for the case of optical symmetry, no attempt was made to analyze the effect of the optical properties of the boundaries on the temperature field in the layer.

These papers can be divided into two groups according to the method of analysis used. The first group includes papers based on the integral equations of radiative transfer, with the corresponding integral analytical methods [1, 2]. Similar in nature are [3, 4] which use the slab method, applicable to electrical-analog computation, as well as a recent paper [8] based on probability methods.

The second group of papers [5–7] is based on the so-called differential methods. Of particular interest is [7], which develops these methods to an advanced degree. In several papers the problem of radiative transfer is analyzed in conjunction with more complex problems (cf., e.g., [10, 11]).

In the present work we shall attempt to carry out an approximate analytical study of problems connected with radiative heat transfer in a plane layer of an absorbing, emitting, nonscattering gray medium with temperature-independent optical properties. The layer is bounded by two parallel, diffusely emitting and diffusely reflecting, isothermal, gray planes.

The paper presents the fundamental formulation of the problem, which consists in: (a) the determination of the net heat flux on the basis of given temperature distribution (direct formulation), and (b) the determination of the temperature distribution on the basis of given distribution of the net radiative heat source per unit volume and boundary temperatures (inverse formulation). The analysis is based on integral methods appropriate to the integral equations which represent the net total and hemispherical radiation flux densities [12].

The integral equations of radiative transfer in a radiating system of arbitrary configuration are [12]

$$E(M) = A(M)\int_F A(N)E_0(N)\Gamma(M, N)\,dF_N + A(M)\int_V \varkappa(P)\eta_0(P)Z(M, P)\,dV_P - A(M)E_0(M) \quad (M \in F), \tag{1}$$

$$-\eta(M) = \varkappa(M)\eta_0(M) - \varkappa(M)\int_V \varkappa(P)\eta_0(P)Z_1(M, P)\,dV_P - \varkappa(M)\int_F A(N)E_0(N)\Gamma_1(M, N)\,dF_N \quad (M \in V),$$

$$(-\eta(M) = \operatorname{div} \mathbf{E}_{4\pi}, \quad \eta_0(M) = 4E_0(M) = 4\sigma_0 T^4(M),$$

$$E_0 = \sigma_0 T^4(M) \quad A(N) = 1 - R(N)). \tag{2}$$

Here $\eta(M)$ is the net total radiative heat source per unit volume at the point M, $\mathbf{E}_{4\pi}$ is the radiation flux vector, E(M) is the net hemispherical radiative heat flux at the point M, $\eta_0(M)$ is the total black-body source function, $E_0(M)$ is the hemispherical black-body radiation flux density, $\varkappa(M)$ is the volumetric absorption (emission) coefficient of the medium at the point M, and A(N) is the emissivity of the surface at the point N. In Eqs. (1) and (2) the resolving kernels Z(M, P), Γ(M, N) have the physical meaning of shape factors between the fixed point M and the generic volume element P and surface element N:

$$Z(M, P) = L(M, P) + \int_F R(N)L(M, N)Z(N, P)\,dF_N,$$

$$\Gamma(M, N) = Q(M, N) + \int_F R(P)Q(M, P)\Gamma(P, N)\,dF_P,$$

$$L(M, P) = \exp\left(-\int_{r^*}^{r} \varkappa(P)\,dS\right)\frac{\cos\theta_M}{\pi r_{MP}^2},$$

$$Q(M, N) = \exp\left(-\int_0^r \varkappa(P)\,dS\right)\frac{\cos\theta_N\cos\theta_M}{\pi r_{MN}^2}.$$

In an analogous manner we can represent the resolving kernels $Z_1(M, P)$, $\Gamma_1(M, N)$ by the equations

$$Z_1(M, P) = L_1(M, P) + \int_F R(N)Q_1(M, N)Z_1(N, P)\,dF_N,$$

$$\Gamma_1(M, N) = Q_1(M, N) + \int_F R(P)Q_1(M, P)\Gamma_1(P, N)\,dF_P,$$

$$L_1(M, P) = \exp\left(-\int_{r^*}^{r} \varkappa(P)\,dS\right)\frac{1}{4\pi r_{MP}^2},$$

$$Q_1(M, N) = \exp\left(-\int_0^r \varkappa(P)\,dS\right)\frac{\cos\theta_N}{\pi r_{MN}^2}.$$

Here r is the distance between M and P or M and N, and θ_M, θ_N are the angles between the generic ray and the normals to the surface elements at M and N.

To apply the integral equations (1), (2) to the specific radiating system under consideration, we formally introduce into these equations the geometrical characteristics of the system. In particular, if we take into account the fact that the surfaces F_1, F_2, which constitute the boundary F, are nonconcave, we find that the self-irradiation shape factors of the surfaces vanish, $Q(P_1, N_1) = Q(P_2, N_2) = 0$ (here P_1 and P_2 are the intermediate reflecting area elements on the surfaces F_1, F_2, respectively), and that reflection takes place between the two outer bounding surfaces only.

One can easily show that the integral terms which appear in (1) and (2) can be expressed in explicit form in terms of the geometrical-optical parameters Q(M, N), Q_1(M, P), L_1(M, P). Thus in Eq. (2) for η(M) we have

$$\int\limits_{F_1} A(N_1)E_0(N_1)\Gamma_1(M, N_1)\,dF_{N_1} = \frac{1}{D_{12}}\int\limits_{F_1} A(N_1)E_0(N_1)\Big[Q_1(M, N_1) + \int\limits_{F_2} R(N_2)Q_1(M, N_2)Q(N_2, N_1)\,dF_{N_2}\Big]dF_{N_1}, \quad (3)$$

$$\int\limits_{F_2} A(N_2)E_0(N_2)\Gamma_1(M, N_2)\,dF_{N_2} = \frac{1}{D_{12}}\int\limits_{F_2} A(N_2)E_0(N_2)\Big[Q_1(M, N_2) + \int\limits_{F_1} R(N_1)Q_1(M, N_1)Q(N_1, N_2)\,dF_{N_1}\Big]dF_{N_2}, \quad (4)$$

$$\int\limits_{V}\varkappa(P)\eta_0(P)Z_1(M, P)\,dV_P = \int\limits_{V}\varkappa(P)\eta_0(P)L_1(M, P)\,dV_P + \frac{1}{D_{12}}\int\limits_{V}\varkappa(P)\eta_0(P)\Big[\int\limits_{F_1} R(N_1)Q_1(M, N_1)Q(N_1, P)\,dF_{N_1} + \int\limits_{F_2} R(N_2)Q_1(M, N_2)Q_1(N_2, P)\,dF_{N_2} + \int\limits_{F_1}\int\limits_{F_2} R(N_1)R(N_2)Q_1(M, N_2)\times Q(N_2, N_1)Q_1(N_1, P)\,dF_{N_2}dF_{N_1} + \int\limits_{F_2}\int\limits_{F_1} R(N_2)R(N_1)Q_1(M, N_1)\times Q(N_1, N_2)Q_1(N_2, P)\,dF_{N_1}dF_{N_2}\Big]dV_P. \quad (5)$$

The effect of multiple reflections is represented here by the geometrical-optical relation

$$D_{12} = 1 - \int\limits_{F_1}\int\limits_{F_2} R(N_1)R(N_2)Q(N_1, N_2)Q(N_2, N_1)\,dF_{N_2}dF_{N_1}.$$

Substituting the above geometrical-optical parameters, analogous to radiation shape factors, into (2), we obtain for η(M) an integral equation in a form which can be used directly in further calculations.

In the course of the calculations we use the exponential integral $K_n(x)$, which is characteristic of transfer processes in absorbing media, the optical depth τ, and the optical thickness τ_0 of the layer,

$$K_n(x) = \int\limits_0^1 e^{-x/\mu}\mu^{n-1}\frac{d\mu}{\mu}, \qquad \tau = \int\limits_0^y \varkappa(\zeta)\,d\zeta, \qquad \tau_0 = \int\limits_0^\delta \varkappa(\zeta)\,d\zeta.$$

In the case when F_1 and F_2 are optically homogeneous and isothermal, the expression for the net radiative heat source per unit volume is

$$-\eta(\tau) = \frac{dE(\tau)}{d\tau} = 4E_0(\tau) - 2\int\limits_0^{\tau_0} E_0(\zeta)K_1|\tau-\zeta|\,d\zeta - \frac{2}{1-4R_1R_2K_3^2(\tau_0)}\Big\{A_1E_{0,1}[K_2(\tau) + 2R_2K_3(\tau_0)K_2(\tau_0-\tau)] + A_2E_{0,2}[K_2(\tau_0-\tau) + 2R_1K_3(\tau_0)K_2(\tau)] + 2R_1K_2(\tau)\int\limits_0^{\tau_0}E_0(\tau)K_2(\tau)\,d\tau + 2R_2K_2(\tau_0-\tau)\int\limits_0^{\tau_0}E_0(\tau)K_2(\tau_0-\tau)\,d\tau + 4R_1R_2K_3(\tau_0)\times\Big[K_2(\tau_0-\tau)\int\limits_0^{\tau_0}E_0(\tau)K_2(\tau)\,d\tau + K_2(\tau)\int\limits_0^{\tau_0}E_0(\tau)K_2(\tau_0-\tau)\,d\tau\Big]\Big\}. \quad (6)$$

This expression agrees with an analogous expression for $\eta(\tau)$ in [13], obtained by a direct radiative heat flux balance.

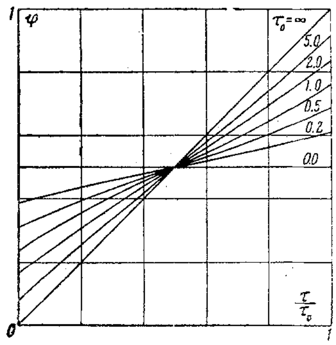

Fig. 1. Distribution of dimensionless temperature $\varphi(\zeta)$ as a function of the optical depth τ/τ_0 of a radiating gas bounded by perfectly black surfaces.

Integrating the right and left sides of (6) term by term, and using the rules for integration under an integral sign, we obtain an expression for the net hemispherical radiative heat flux. Following [14], we represent the latter, as well as Eq. (6), in the form

$$E(\tau) = 2A_1(\tau)E_{0,1} - 2A_2(\tau)E_{0,2} + 2\int\limits_0^{\tau} E_0(\zeta)\Psi(\tau, \zeta)\,d\zeta - 2\int\limits_{\tau}^{\tau_0} E_0(\zeta)\Psi(\zeta, \tau)\,d\zeta,$$

$$\frac{dE(\tau)}{d\tau} = 4E_0(\tau) - 2A_1'(\tau)E_{0,1} - 2A_2'(\tau)E_{0,2} - 2\int\limits_0^{\tau_0} E_0(\zeta)Z(\tau, \zeta)\,d\zeta, \quad (7)$$

which contains the geometrical-optical parameters

$$A_1(\tau) = \frac{A_1}{D_{12}}[K_3(\tau) - 2R_2K_3(\tau_0)K_3(\tau_0-\tau)],$$

$$A_2(\tau) = \frac{A_2}{D_{12}}[K_3(\tau_0-\tau) - 2R_1K_3(\tau_0)K_3(\tau)],$$

$$A_1'(\tau) = \frac{A_1}{D_{12}}[K_2(\tau) + 2R_2K_3(\tau_0)K_2(\tau_0-\tau)],$$

$$A_2'(\tau) = \frac{A_2}{D_{12}}[K_2(\tau_0-\tau) + 2R_1K_3(\tau_0)K_2(\tau)],$$

$$D_{12} = 1 - 4R_1R_2K_3^2(\tau_0),$$

$$\Psi(\tau,\zeta) = K_2(\tau-\zeta) + 2A_1(\tau)\frac{R_1}{A_1}\times$$

$$\times K_2(\zeta) - 2A_2(\tau)\frac{R_2}{A_2}K_2(\tau_0-\zeta),$$

$$\Psi(\zeta,\tau) = K_2(\zeta-\tau) - 2A_1(\tau)\frac{R_1}{A_1}\times$$

$$\times K_2(\zeta) + 2A_2(\tau)\frac{R_2}{A_2}K_2(\tau_0-\zeta),$$

$$Z(\tau,\zeta) = K_1|\tau-\zeta| + 2A_1'(\tau)\frac{R_1}{A_1}\times$$

$$\times K_2(\zeta) + 2A_2'(\tau)\frac{R_2}{A_2}K_2(\tau_0-\zeta). \tag{8}$$

When the radiating system is in thermodynamic equilibrium, equations (7) and (8) degenerate into the integral equations for a closed system

$$A_1(\tau) - A_2(\tau) + \int_0^\tau \Psi(\tau,\zeta)\,d\zeta - \int_\tau^{\tau_0}\Psi(\zeta,\tau)\,d\zeta = 0,$$
$$A_1'(\tau) + A_2'(\tau) + \int_0^{\tau_0} Z(\tau,\zeta)\,d\zeta = 2. \tag{9}$$

Using (9), we can represent (7) and (8) in the dimensionless form

$$q = 2A_2(\tau) - 2\int_0^\tau \Psi(\tau,\zeta)\varphi(\zeta)\,d\zeta + 2\int_\tau^{\tau_0}\Psi(\zeta,\tau)\varphi(\zeta)\,d\zeta, \tag{10}$$

$$\frac{dq(\tau)}{d\tau} = 4\varphi(\tau) - 2A_2'(\tau) - 2\int_0^{\tau_0} Z(\tau,\zeta)\varphi(\zeta)\,d\zeta$$
$$\left(q(\tau) = \frac{E(\tau)}{E_{0,1}-E_{0,2}},\quad \varphi(\tau) = \frac{E_0(\tau)-E_{0,1}}{E_{0,2}-E_{0,1}}\right). \tag{11}$$

The analysis of heat transfer in radiative equilibrium, when the net total radiative heat source per unit volume is identically equal to zero and the net hemispherical radiative heat flux is constant ($E(\tau) =$ const, $dE/d\tau \equiv 0$), reduces to the simultaneous solution of the equations

$$\varphi(\tau) = \frac{1}{2}\left(A_2'(\tau) + \int_0^{\tau_0} Z(\tau,\zeta)\varphi(\zeta)\,d\zeta\right), \tag{12}$$

$$q = 2\left(A_2(0) + \int_0^{\tau_0}\Psi(\tau,\zeta)\varphi(\zeta)\,d\zeta\right), \tag{13}$$

which follow from (10), (11). Note that the calculation of the dimensionless heat flux q is reduced to a quadrature of $\varphi(\tau)$, which is obtained from the solution of (12). Thus, essentially, the whole problem is reduced to the analysis and solution of Eq. (12), which is a Fredholm equation of the second kind with the kernel $K_1|\tau-\zeta|$, which has a logarithmic singularity at $\tau = \zeta$.

Note, that when $A_1 = A_2$, the condition of optical symmetry of the radiating system yields the obvious relation

$$\varphi(\tau) + \varphi(\tau_0-\tau) = 1, \quad \text{or} \quad \varphi(^1/_2\tau_0) + \varphi(^1/_2\tau_0) = 1, \quad \varphi(^1/_2\tau_0) = {}^1/_2.$$

On the other hand, in the general case Eq. (12) obviously yields

$$\varphi(\tau)|_{\tau_0=0} = \frac{1}{2}\left(1 + \frac{R_1-R_2}{1-R_1R_2}\right).$$

In the case of optical symmetry ($R_1 \equiv R_2$), $\varphi(\tau) = {}^1/_2$ for $\tau_0 = 0$. One should note the generalized form of the integral equation (12), which is characteristic of transfer processes which involve the notion of a mean free path.

In particular, in the analysis of internal shear of rarefied gases one obtains an integral equation which describes the distribution of the mean velocity of a gas bounded by two parallel planes, one of which is moving, which in the case of inelastic molecules takes on the form [15]

$$w(\tau) = \frac{1}{2}\left\{w_0K_2(\tau_0-\tau) + \int_0^{\tau_0}K_1|\tau-\zeta|w(\zeta)\,d\zeta\right\}$$
$$[\tau = \sigma y,\ \tau_0 = \sigma h,\ \sigma = 1/l].$$

Here l is the molecular mean free path, w_0 is the velocity of the moving boundary, $w(\tau)$ is the mean velocity of the gas at the distance τ away from the wall, and h is the thickness of the gas layer. This equation is identical to the integral equation

$$E_0(\tau) = \frac{1}{2}\left\{E_{0,2}K_2(\tau_0-\tau) + \int_0^{\tau_0}K_1|\tau-\zeta|E_0(\zeta)\,d\zeta\right\},$$

which is a particular case of (12), written in dimensional form, for the case when the bounding walls are perfectly black and the emissive power of one of these is zero.

The analysis of the molecular transfer in a plane layer of a rarefied gas indicates that the profile of the mean velocity is linear over a considerable portion of gas layer [15]. Thus, as a first approximation we may use a linear dependence of $\varphi(\tau)$ on the optical depth τ

$$\varphi(\tau) = \varphi(0) + \frac{\varphi(\tau_0)-\varphi(0)}{\tau_0}\tau. \tag{14}$$

Substituting (14) into the integral equation (12), we obtain the second approximation

$$\varphi(\tau) = a(\tau) + b(\tau)\varphi(0) + c(\tau)(\varphi(\tau_0) - \varphi(0)), \tag{15}$$

where

$$a(\tau)=\frac{1}{2}A_2'(\tau),\quad b(\tau)=\frac{1}{2}\Big[2-K_2(\tau)-K_2(\tau_0-\tau)+$$
$$+2\left(\frac{1}{2}-K_3(\tau_0)\right)\left(A_1'(\tau)\frac{R_1}{A_1}+A_2'(\tau)\frac{R_2}{A_2}\right)\Big],$$
$$c(\tau)=\frac{1}{2\tau_0}\Big[2\tau+K_3(\tau)-K_3(\tau_0-\tau)-\tau_0K_2(\tau_0-\tau)+$$
$$+2\left(\frac{1}{3}-K_4(\tau_0)\right)\left(A_1'(\tau)\frac{R_1}{A_1}-A_2'(\tau)\frac{R_2}{A_2}\right)\Big]-$$
$$-A_1'(\tau)\frac{R_1}{A_1}K_3(\tau_0)+\frac{1}{2}A_2'(\tau)\frac{R_2}{A_2}. \tag{16}$$

Evaluating (15) at $\tau = 0$ and at $\tau = \tau_0$, we obtain a system of algebraic equations for $\varphi(0)$ and $\varphi(\tau_0)$, which has the solution

$$\varphi(0)=\frac{a(0)(1-c(\tau_0))+a(\tau_0)c(0)}{1-b(0)-c(\tau_0)(1-b(0))+c(0)(1-b(\tau_0))}$$
$$\varphi(\tau_0)=\frac{a(\tau_0)(1-b(0))-a(0)(c(\tau_0)-a(\tau_0)-b(\tau_0))}{1-b(0)-c(\tau_0)(1-b(0))+c(0)(1-b(\tau_0))}. \tag{17}$$

Here $a(0)$, $a(\tau_0)$, $b(0)$, $b(\tau_0)$, $c(0)$, and $c(\tau_0)$ are the values of a, b, c [Eq. (16)] at $\tau = 0$ and $\tau = \tau_0$.

When the radiating system is optically symmetrical ($R_1 = R_2$), the solution of (12) with the linear approximation can be written

$$\varphi(\tau)=\frac{1}{D(\tau_0)}\Big\{\frac{1}{2}-K_3(\tau_0)+2\left(\frac{1}{3}-K_4(\tau_0)\right)\times$$
$$\times\left(A_1'(0)\frac{R_1}{A_1}-A_2'(0)\frac{R_2}{A_2}\right)+$$
$$+\tau\Big[1-K_2(\tau_0)-2\left(\frac{1}{2}+K_3(\tau_0)\right)\left(A_1'(0)\frac{R_1}{A_1}-A_2'(0)\frac{R_2}{A_2}\right)\Big]\Big\},$$
$$D(\tau_0)=1+\tau_0(1-K_2(\tau_0))-2K_3(\tau_0)+ \tag{18}$$
$$+\left[\frac{4}{3}-\tau_0(1+2K_3(\tau_0)-4K_4(\tau_0))\right]\left(A_1'(0)\frac{R_1}{A_1}-A_2'(0)\frac{R_2}{A_2}\right).$$

In the special case when the bounding walls are perfectly black ($R_1 = R_2 = 0$), solution (18) takes on the much simpler form

$$\varphi(\tau)=\frac{1/2-K_3(\tau_0)+\tau(1-K_2(\tau_0))}{1+\tau_0(1-K_2(\tau_0))-2K_3(\tau_0)}, \tag{19}$$

and solution (15) becomes

$$\varphi(\tau)=\frac{1}{2D(\tau_0)}\Big\{1-2K_3(\tau_0)+$$
$$+(1-K_2(\tau_0))(2\tau+K_3(\tau)-K_3(\tau_0-\tau))-$$
$$-\left(\frac{1}{2}-K_3(\tau_0)\right)\left(K_2(\tau)-K_2(\tau_0-\tau)\right)\Big\}$$
$$[D(\tau_0)=1+\tau_0(1-K_2(\tau_0))-2K_3(\tau_0)]. \tag{20}$$

The distribution

$$\varphi(\zeta)=\frac{E_0(\zeta)-E_{0\cdot1}}{E_{0.2}-E_{0\cdot1}},\ \zeta=\frac{\tau}{\tau_0}$$

as a function of the optical depth in a radiating gas (Fig. 1) for $\tau_0 = 0$, 0.2, 0.5, 1.0, 2.0, 5.0, ∞, calculated from Eq. (20), is in very good agreement (within <1%) with the exact numerical solution given in [2]. It can be seen that the linear approximation is in satisfactory (within <3% error) agreement with the second approximation to the solution of (12).

The temperature distribution in a plane layer of a radiating gas is determined, as can be seen from (14) (17), and (18), by the optical thickness of the layer and by the optical properties of the radiating surfaces.

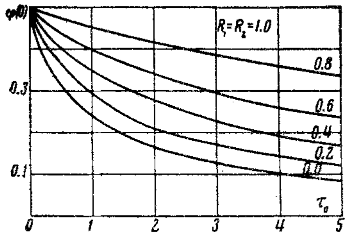

Fig. 2. Dimensionless temperature at the wall $\tau \approx 0$, $\varphi(0)$, as a function of the optical thickness of the gas layer.

Figure 2 shows the dimensionless temperature near the wall $\tau = 0$, $\varphi(0)$, which represents the temperature slip, as a function of the optical thickness τ_0 for the case of optical symmetry for the values $R_1 = R_2 = R = 0$, 0.2, 0.4, 0.6, 0.8 and 1.0.

As might be expected, the temperature slip $\varphi(0)$ increases with increasing R and attains its maximum value $\varphi(0) = 1/2$, which is independent of τ_0, for $R = 1.0$.* In the absence of optical symmetry ($R_1 \neq R_2$), the inflection point, which is characteristic for the case with optical symmetry (Fig. 1), moves towards the wall with the higher emissivity. Figure 3 shows the distribution $\varphi(\zeta)$, based on the linear approximation in Eqs. (14) and (18), for $R_1 = 0.3$ and $R_2 = 0.8$ for the values $\tau_0 = 0$, 0.2, 0.5, 1.0, 3.0, 5.0 and ∞.

Generalizing our consideration ot the approximate solution of the integral equation (12), we note that such equations have unbounded kernels of linear-potential type, for which Fredholm's theorems in the plane of the complex parameter $\lambda = 1/2$ hold the same way as they hold for continuous kernels. This allows us to solve integral equations of the type of equation (12) by the usual methods and, in particular, by the interation method.

The solution of the particular form of (12) corresponding to perfectly black surfaces,

$$\varphi(\tau)=\frac{1}{2}\left(K_2(\tau_0-\tau)+\int_0^{\tau_0}K_1|\tau-\zeta|\varphi(\zeta)d\zeta\right), \tag{21}$$

can be obtained in the following manner.

*Rigorously speaking, the case $R \equiv 1.0$ should be excluded from our consideration, since it does not guarantee the uniqueness of the solution of the integral equations.

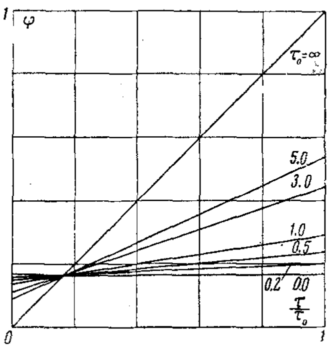

Fig. 3. Distribution of $\varphi(\zeta)$ as a function of the optical depth τ/τ_0 in a gas layer bounded by reflecting surfaces with $R_1 = 0.3$ and $R_2 = 0.8$.

Following [17], we differentiate (21) with respect to τ. Differentiating under the integral sign we use the obvious property of the kernel $K_1|\tau - \zeta|$, which can be written in the form of the conservation equation

$$\frac{\partial K_1|\tau-\zeta|}{\partial\tau} + \frac{\partial K_1|\tau-\zeta|}{\partial\zeta} = 0.$$

After some simple transformations, we obtain

$$\varphi'(\tau) = \frac{1}{2}(K_1(\tau) + K_1(\tau_0-\tau))\varphi(0) + \frac{1}{2}\int_0^{\tau_0} K_1|\tau-\zeta|\varphi'(\zeta)\,d\zeta. \quad (22)$$

Solving (22) by means of resolving kernels and integrating the solution from 0 to τ, we obtain

$$\varphi(\tau) = \varphi(0) + \frac{1}{2}\varphi(0)\left(\int_0^{\tau}\Gamma_1(\zeta)\,d\zeta + \int_0^{\tau}\Gamma_1(\tau_0-\zeta)\,d\zeta\right). \quad (23)$$

Assuming here $\tau = \tau_0/2$ and taking into account that $\varphi(\tau_0/2) = 1/2$, we have

$$\varphi(0) = \left(2 + \int_0^{\tau_0}\Gamma_1(\tau)\,d\tau\right)^{-1}. \quad (24)$$

The general solution of (21) can be finally written in the generalized form

$$\varphi(\tau) = \left(1 + \frac{1}{2}\int_0^{\tau}\Gamma_1(\zeta)\,d\zeta + \frac{1}{2}\int_0^{\tau}\Gamma_1(\tau_0-\zeta)\,d\zeta\right) \times \left(2 + \int_0^{\tau_0}\Gamma_1(\tau)\,d\tau\right)^{-1},$$

$$\Gamma_1(\tau) = K_1(\tau) + \frac{1}{2}\int_0^{\tau_0} K_1|\tau-\zeta|\Gamma_1(\zeta)\,d\zeta, \quad (25)$$

$$\Gamma_1(\tau_0-\tau) = K_1(\tau_0-\tau) + \frac{1}{2}\int_0^{\tau_0} K_1|\tau-\zeta|\Gamma(\tau_0-\zeta)\,d\zeta.$$

The good convergence of the iteration process allows us to use (25) with approximate expressions for the resolving kernels (the series of iterated kernels may be truncated at the second term).

Note, that comparing the expressions for $\varphi(0)$ from (19) and (24) we obtain the rigorously exact integral identity

$$\int_0^{\tau_0}\Gamma_1(\tau)\,d\tau = \tau_0\frac{1-K_2(\tau_0)}{1/2 - K_3(\tau_0)}.$$

From Eq. (13) for the net hemispherical radiation flux it follows that the latter can be calculated by quadratures of the temperature distribution.

To calculate the integral on the right side of (13), we use the linear approximation for $\varphi(\tau)$ in the form (14), where $\varphi(0)$ and $\varphi(\tau_0)$ are determined either from the general solution (17), or from the particular solution (18) obtained for the case of optical symmetry. After several transformations, the general expression for the dimensionless flux can be written in the form

$$q = 2A_2(0) + 2\varphi(0)(A_1(0) - A_2(0)) - 2\frac{\varphi(\tau_0)-\varphi(0)}{1-4R_1R_2K_3^2(\tau_0)} \times \left\{A_1A_2K_3(\tau_0) - \frac{1}{\tau_0}A_1(1-2R_2K_3(\tau_0))(1/3 - K_4(\tau_0))\right\}.$$

$$A_1(0) = A_1\frac{1/2 - 2R_2K_3(\tau_0)}{1-4R_1R_2K_3^2(\tau_0)}, \quad A_2(0) = \frac{A_1A_2K_3(\tau_0)}{1-4R_1R_2K_3^2(\tau_0)}. \quad (26)$$

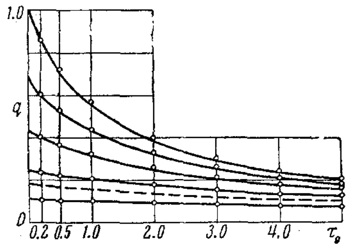

Fig. 4. Dimensional heat flux q as a function of the optical thickness τ_0.

Here we have used the equation of a closed system (9), which for $\tau = 0$ yields

$$A_1(0) - A_2(0) - \int_0^{\tau_0}\Psi(\zeta)\,d\zeta = 0.$$

In the case of optical symmetry one should take into account that

$$\varphi(\tau_0) = 1 - \varphi(0).$$

Of particular interest is the particular case of (26) corresponding to perfectly black surfaces. The dimensionless flux takes then the relatively simple form

$$q = \left(\frac{1}{2} + K_3(\tau_0) + \frac{2}{3}(1-K_2(\tau_0))\frac{1-3K_4(\tau_0)}{1-2K_3(\tau_0)}\right) \times \left(1 + \tau_0\frac{1-K_2(\tau_0)}{1-2K_3(\tau_0)}\right)^{-1}. \quad (27)$$

Calculations based on (27) are practically identical with the result of exact numerical solutions given in [2]. Figure 4 shows q as a function of the optical thickness of the layer, calculated from Eq. (26) for the values $R_1 = R_2 = R = 0, 0.2, 0.4, 0.6$, and 0.8. The small circles represent values calculated from the approximate formula

$$q = (1/A_1 + 1/A_2 - 1 + {}^3/_4\,\tau_0)^{-1}, \qquad (28)$$

which has been obtained by the differential method, assuming isotropic intensity distribution throughout the layer [7].

As could be expected, the agreement between the simplified formula (28) and formula (26) increases with increasing R. The dashed line in Fig. 4 represents the results for q for the case of optical asymmetry ($R_1 = 0.3$, $R_2 = 0.8$), which are also in good agreement with (28).

In Conclusion, one should note the conservative nature of q with respect to the imposed temperature distribution.

In this paper we have demonstrated the continuous transition from the general equations of radiative transfer (1), (2) to the particular form (7), (8). We have constructed the integral equations (12), (13), whose form is characteristic of transfer processes characterized by the notion of a mean free path.

We have shown the physical motivation for the use of the linear approximation (14)–(17) for $\varphi(\tau)$. Using this approximation to obtain the second approximation (20), which practically coincides with the exact solution, we have demonstrated the relatively fast convergence of the iterative solution of (12).

In the case of optical symmetry of the radiating system, the temperature distribution $\varphi(\zeta)$ has a fixed inflection point at the midplane, (independent of the optical thickness of the layer). The temperature slip at the walls is determined by the optical thickness of the layer τ_0 and by the optical properties of the surfaces R. The temperature slip decreases with increasing τ_0 and increases with increasing R.

In the absence of optical symmetry the inflection point moves away from the midplane in the direction of the wall with the higher emissivity. The values of the net hemispherical radiation flux density calculated from (26) and (27) with the linear temperature distribution, coincide with the results of exact numerical calculations.

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